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# Zeros in the complex $\beta$-plane of 2D Ising block spin Boltzmannians

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**Abstract.** Effective Boltzmannians in the sense of the block spin renormalization group are computed for the 2D Ising model. The blocking is done with majority and Kadanoff rules for blocks of size $2 \times 2$. Transfer matrix techniques allow the determination of effective Boltzmannians as polynomials in $u = \exp(4\beta)$ for lattices of up to $4 \times 4$ blocks. The zeros of these polynomials are computed for all non-equivalent block spin configurations. Their distribution in the complex $\beta$-plane reflects the regularity structure of the block spin transformation. In the case of the Kadanoff rule spurious zeros approach the positive real $\beta$-axis at large values of $\beta$. They might be related to the renormalization group pathologies discussed in the literature.

## 1. Introduction

Regularity is at the heart of position space (block spin) renormalization group. It is usually assumed, and is of central importance, that coupling constants of the block spin effective Hamiltonian depend in a non-singular way on the parameters of the original theory. There are, however, situations where this assumption is not valid. Pathological behaviour in renormalization groups of the low-temperature Ising model was first observed by Israel [1] and Griffith and Pearce [2, 3]. An extensive and rigorous analysis of regularity properties and pathologies in Ising model block spin transformations was performed by van Enter *et al* [4]. The central observation is that in certain situations the effective measure for the block spin theory cannot be represented as $\exp(-H)$. This means that the effective measure is non-Gibbsian. (See also [5] for a careful analysis of the situation.)

This paper presents some numerical results on the distribution of zeros in the complex $\beta$-plane for block spin Boltzmannians of the 2D Ising model. These Boltzmannians are partition functions with 'fixed' block spins $\mu$, namely

$$B(\mu) = \sum_{\sigma} P(\mu, \sigma) \exp[-\beta H(\sigma)]. \tag{1}$$

$P(\mu, \sigma)$ encodes the blocking rule. Why should these zeros provide interesting information? If the usual renormalization group assumptions are true, the zeros of $B(\mu)$ should—for all block spin configurations $\mu$—behave 'better' than those of the full partition function. Note that the zeros of the full partition function approach the real axis at the critical point [6]. This should *not* happen for the zeros of $B(\mu)$! Furthermore, one might expect that the

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pathologies described in the literature are related to the distribution of zeros close to the $\beta$-axis at large positive values.

This article is organized as follows. In section 2 the model notation is set up, and the blocking rules are defined. Section 3 gives a sketch of the transfer matrix technique used to compute the polynomials. Section 4 summarizes the numerical results for the majority rule blocking. Observations for the Kadanoff blocking rule are reported in section 5. Conclusions follow.

## 2. Model and block spin definition

We deal with the 2D Ising model, with partition function

$$Z = \sum_{\sigma} \exp[-\beta H(\sigma)] \tag{2}$$

where

$$H(\sigma) = -\sum_{\langle i,j\rangle} \sigma_i \sigma_j . \tag{3}$$

The $\sigma_i$ assume values $\pm 1$ and are defined on a square lattice of extension $L \times L$, supplied with periodic boundary conditions. The energy $H$ is a sum over all pairs of nearest neighbours. In the infinite volume limit the model undergoes a second-order phase transition at $\beta_c = \frac{1}{2}\ln(\sqrt{2}+1) = 0.440\,6868$. A block spin transformation with scale factor 2 is defined as follows. For $L$ even, the lattice is divided in blocks of size $2 \times 2$. Given the configuration of the $\sigma$-spins in a block $I$, a block spin $\mu_I$ is chosen with probability $P(\mu, \sigma)$. The majority rule is defined through

$$P(\mu, \sigma) = \prod_{\text{blocks } I} p_I(\mu_I, \sigma) \tag{4}$$

with

$$p_I(\mu_I, \sigma) = \begin{cases} \frac{1}{2} & \text{if } \sum_{i \in I} \sigma_i = 0 \\ 1 & \text{if } \mu_I \sum_{i \in I} \sigma_i > 0 \\ 0 & \text{else.} \end{cases} \tag{5}$$

The so-called Kadanoff rule is

$$p_I(\mu_I, \sigma) = \frac{\exp(\omega \mu_I \sum_{i \in I} \sigma_i)}{2\cosh(\omega \sum_{i \in I} \sigma_i)} . \tag{6}$$

In the limit $\omega \to \infty$ one recovers the majority rule.

Given that $\sum_{\mu} P(\mu, \sigma) = 1$, the full partition function can be rewritten as

$$Z = \sum_{\mu} B(\mu) \tag{7}$$

with

$$B(\mu) = \sum_{\sigma} P(\mu, \sigma) \exp[-\beta H(\sigma)] . \tag{8}$$

Usually, one aims at a parametrization $B(\mu) = \exp[-H'(\mu)]$, where $H'$ is the effective Hamiltonian. Note, however, that this is impossible in the pathological situations discussed in the literature [4].

**Table 1.** Notation for transfer matrix in the case $L = 6$.

| | | |
|---|---|---|
| $\tau_.$ $\tau_.$<br>$\boxed{\mu_7}$<br>$\sigma_.$ $\sigma_.$ | $\tau_.$ $\tau_.$<br>$\boxed{\mu_8}$<br>$\sigma_.$ $\sigma_.$ | $\tau_.$ $\tau_.$<br>$\boxed{\mu_9}$<br>$\sigma_.$ $\sigma_.$ |
| $\tau_.$ $\tau_.$<br>$\boxed{\mu_4}$<br>$\sigma_.$ $\sigma_.$ | $\tau_.$ $\tau_.$<br>$\boxed{\mu_5}$<br>$\sigma_.$ $\sigma_.$ | $\tau_.$ $\tau_.$<br>$\boxed{\mu_6}$<br>$\sigma_.$ $\sigma_.$ |
| $\tau_1$ $\tau_2$<br>$\boxed{\mu_1}$<br>$\sigma_1$ $\sigma_2$ | $\tau_3$ $\tau_4$<br>$\boxed{\mu_2}$<br>$\sigma_3$ $\sigma_4$ | $\tau_5$ $\tau_6$<br>$\boxed{\mu_3}$<br>$\sigma_5$ $\sigma_6$ |

## 3. Computation of $\boldsymbol{B(\mu)}$

An exact computation of $B(\mu)$ as function of $\beta$ seems impossible. However, on lattices up to at least $L = 8$ it can be determined by numerical transfer matrix calculations. $L = 8$ corresponds to $4 \times 4$ blocks, with $2^{16} = 65\,536$ block spin configurations $\mu$. We do not have to consider them all. Configurations connected through global spin-flip or geometric symmetries (reflections, shifts, rotations) have the same effective Boltzmannian $B(\mu)$. A careful counting yields for the number of non-equivalent configurations $N(L)$

| $L$ | 4 | 6 | 8 | 10 |
|---|---|---|---|---|
| $N(L)$ | 4 | 13 | 479 | 86056 |

The result for $N(10)$ was taken from [7].

After multiplication with a constant prefactor, $B(\mu)$ can be expressed as a polynomial of order $L^2$ in $u = \exp(4\beta)$. The coefficients of this polynomial can be computed by transfer matrix multiplication. In order to avoid notational complication, I give a sketch of the method for the case of $3 \times 3$ blocks ($L = 6$). The generalization to other values of $L$ is then obvious. Some of the notation is depicted in table 1. The effective Boltzmannian can be expressed as

$$B(\mu) = \mathrm{Tr}[\boldsymbol{T} \cdot \boldsymbol{S}(\mu_7, \mu_8, \mu_9) \cdot \boldsymbol{T} \cdot \boldsymbol{S}(\mu_4, \mu_5, \mu_6) \cdot \boldsymbol{T} \cdot \boldsymbol{S}(\mu_1, \mu_2, \mu_3)]. \tag{9}$$

Here, $\boldsymbol{S}$ is a $2^6 \times 2^6$ matrix, labelled by the Ising row configurations. It depends explicitly on the line configuration of prescribed block spins. For example, the matrix elements of $\boldsymbol{S}(\mu_1, \mu_2, \mu_3)$ are

$$\boldsymbol{S}(\mu_1, \mu_2, \mu_3)_{\sigma,\tau} = \exp\left[\beta \sum_{i=1}^{6} (\sigma_i \sigma_{i+1} + \tau_i \tau_{i+1} + \sigma_i \tau_i)\right] \prod_{I=1}^{3} p_I(\mu_I, \sigma, \tau). \tag{10}$$

The matrix $\boldsymbol{T}$, which is also of size $2^6 \times 2^6$, is defined by

$$\boldsymbol{T}_{\sigma,\tau} = \exp\left[\beta \sum_{i=1}^{6} \sigma_i \tau_i\right]. \tag{11}$$

It is not difficult to represent the transfer matrix multiplications in equation (9) in terms of operations on the coefficients of polynomials in $u$. The computer implementation of these operations form the basis for the results presented in this paper.

**Table 2.** Coefficients in the polynomials $B(\mu) = \sum_{k=0}^{L^2} B_k(\mu)u^k$, for the four independent $2 \times 2$ block spin configurations on an $L = 4$ lattice, majority rule.

| $k$ | c#1 $\begin{matrix} + & - \\ - & + \end{matrix}$ | c#2 $\begin{matrix} + & + \\ - & - \end{matrix}$ | c#3 $\begin{matrix} + & + \\ + & - \end{matrix}$ | c#4 $\begin{matrix} + & + \\ + & + \end{matrix}$ |
|---|---|---|---|---|
| 0 | 2 | 2 | 2 | 2 |
| 1 | 0 | 0 | 0 | 0 |
| 2 | 32 | 32 | 32 | 32 |
| 3 | 96 | 64 | 64 | 32 |
| 4 | 544 | 384 | 416 | 416 |
| 5 | 2 336 | 1 728 | 1 728 | 1 120 |
| 6 | 9 360 | 6 336 | 6 560 | 5 232 |
| 7 | 19 712 | 12 960 | 13 344 | 9 536 |
| 8 | 23 674 | 20 906 | 20 570 | 16 426 |
| 9 | 8 224 | 14 592 | 14 112 | 14 688 |
| 10 | 1 456 | 7 360 | 6 720 | 10 448 |
| 11 | 96 | 1 120 | 1 696 | 4 704 |
| 12 | 4 | 52 | 260 | 2 244 |
| 13 | 0 | 0 | 32 | 384 |
| 14 | 0 | 0 | 0 | 256 |
| 15 | 0 | 0 | 0 | 0 |
| 16 | 0 | 0 | 0 | 16 |

## 4. Results for the majority rule

Let us start with some results for $L = 4$. The four non-equivalent configurations, called $c\#1 \ldots c\#4$, are specified in the head of table 2. In the columns we quote the coefficients $B_k(\mu)$ of the polynomial $B(\mu) = \sum_{k=0}^{L^2} B_k(\mu)u^k$.

The zeros of these polynomials were determined with the help of the computer algebra program MapleV. For zeros $u_0$ not lying on the negative real axis, we then computed the corresponding $\beta_0$-values through $\beta_0 = \frac{1}{4}\ln(u_0)$. The distributions of these numbers for $L = 4$ are shown in figure 1, with different symbol code for the four block spin configurations. The figure also contains a circle of radius $\beta_c$ around the origin. Note that the two zeros closest to the critical point belong to configuration $c\#4$.

The results for the $3 \times 3$ block lattice are plotted in figure 2. The 13 non-equivalent block spin configurations are specified in table 3. One observes again that the zeros closest to the critical point belong to the fully magnetized configuration ($c\#13$).

The zeros of the 479 effective Boltzmannians on the $4 \times 4$ block lattice are shown in figure 3. The plot also shows (with crosses) the zeros of the full partition function. It seems that the Boltzmannian zeros do not approach the real axis in the critical region, whereas the full partition function zeros do. To check this in more detail, we compare the distribution of zeros in the critical region with the three available lattice sizes together, see figure 4. The plot clearly demonstrates that the zeros do not move towards the real axis in the critical region. One might conclude from this plot that there should exist in the $L \to \infty$ limit a strip around the real $\beta$-axis ranging from $\beta = 0$ at least up to $\beta_c$ where the effective Boltzmannian is free of $\beta$-zeros. In this region it should thus be possible to take the logarithm without danger. Furthermore, high-temperature (small $\beta$) expansions for the renormalization group could by analytical continuation be used in the critical region.

There is another observation to be made when comparing figures 2 and 3. With

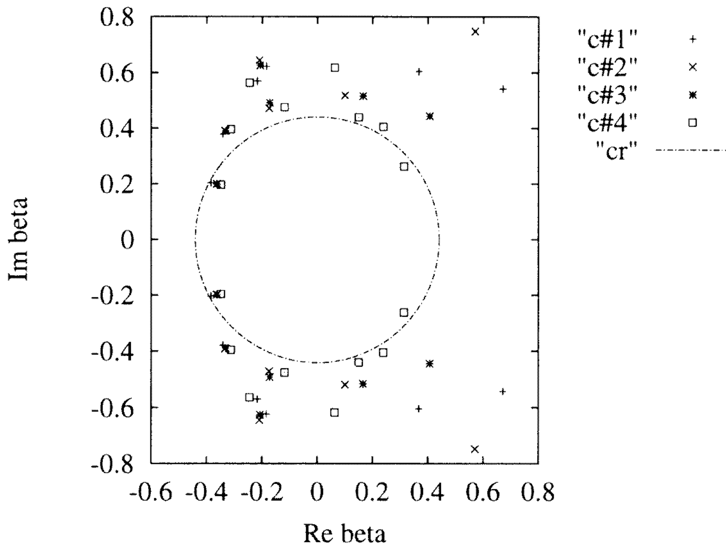


**Figure 1.** Zeros of $B(\mu)$ for the four non-equivalent block spin configurations on a $2 \times 2$ block lattice, majority rule. The circle has radius $\beta_c$.

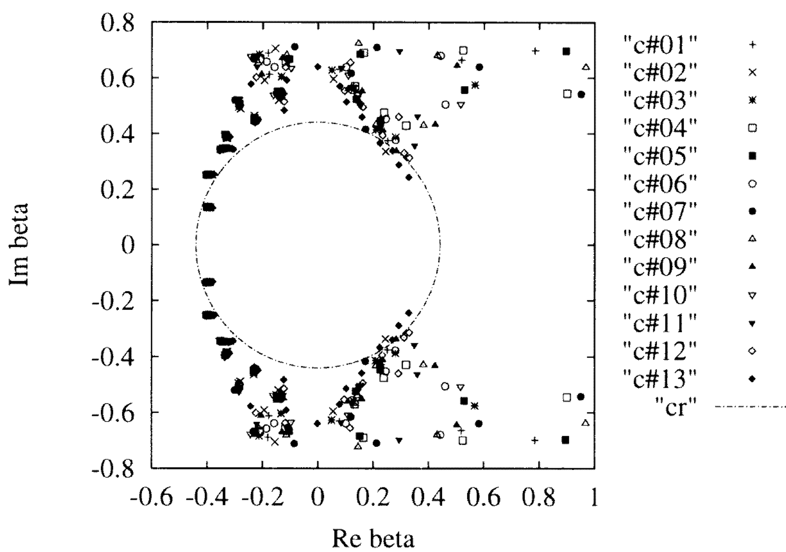


**Figure 2.** Zeros of $B(\mu)$ for the 13 non-equivalent block spin configurations on a $3 \times 3$ block lattice, majority rule.

increasing $L$, more and more zeros populate the part of the plane with larger real part of $\beta$. They are not obviously approaching the real axis there, but we also cannot exclude such a scenario. Note that in the analysis of van Enter *et al* [4] the case of the 2D Ising majority rule was not treated for $L = 2$. It is therefore, at the moment, not clear whether, in this case, a large $\beta$-pathology exists. We shall see in the next section that in case of the Kadanoff rule (where pathologies do exist) spurious zeros seem to approach the axis at large positive $\beta$.

**Table 3.** The 13 non-equivalent block spin configurations on a $3 \times 3$ lattice.

| $c$#01 | $c$#02 | $c$#03 | $c$#04 |
|---|---|---|---|
| + + − | + + − | + + + | + − + |
| + + − | + + − | + − − | + + − |
| + − − | + + − | + − − | + − − |
| $c$#05 | $c$#06 | $c$#07 | $c$#08 |
| − + + | + + + | − − + | + − + |
| + + − | + + − | + + − | + + − |
| + − − | + − − | + + − | + + − |
| $c$#09 | $c$#10 | $c$#11 | $c$#12 |
| + + + | − + + | + + + | + + + |
| + + − | + − + | + − + | + + + |
| + + − | + + − | + + − | + + − |
| $c$#13 | | | |
| + + + | | | |
| + + + | | | |
| + + + | | | |

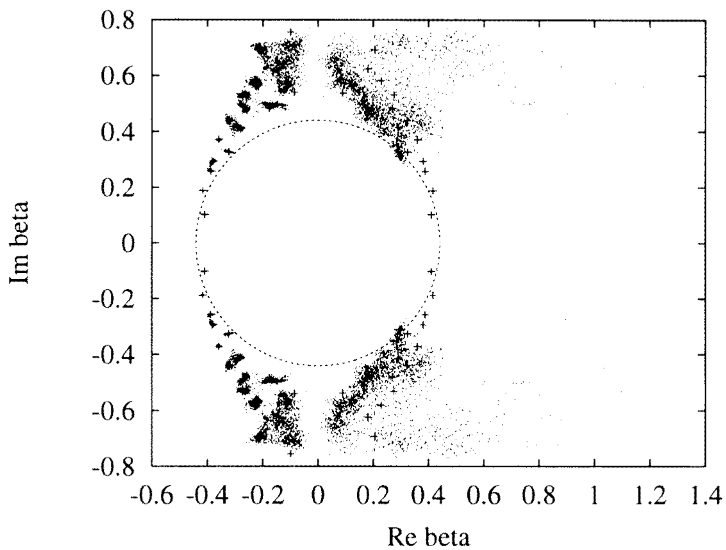


**Figure 3.** Zeros of $B(\mu)$ for the 479 non-equivalent block spin configurations on a $4 \times 4$ block lattice, majority rule. In addition, the zeros of the full partition function on an $8 \times 8$ lattice are shown (crosses).

## 5. Results for Kadanoff rule

It has often been observed that passing from a $\delta$-function block rule to a 'Gaussian smeared' rule improves the locality and analyticity properties of the effective Hamiltonian, see e.g.

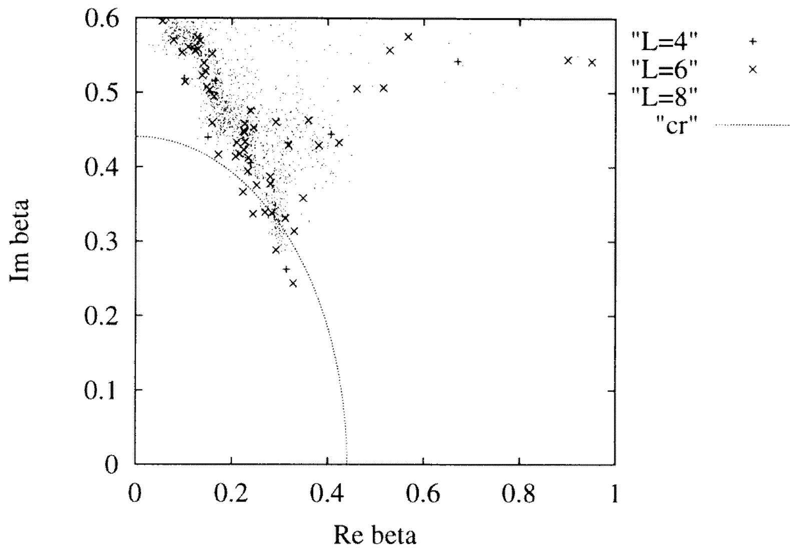


**Figure 4.** Common plot of the Boltzmann zeros for $L = 4$, 6, and 8, majority rule.

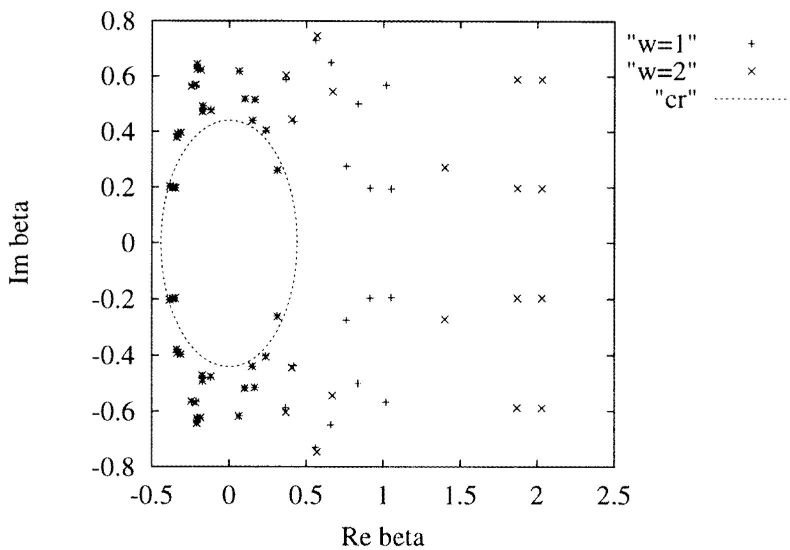


**Figure 5.** Zeros of $B(\mu)$ for $L = 4$, for the majority rule ($\omega = \infty$), and for the Kadanoff rules with $\omega = 1$ and $\omega = 2$.

[8]. In this section some results will be presented on the distribution of zeros in the case of the Kadanoff rule equation (6) for $\omega = 1$ and $\omega = 2$. For finite $\omega$ there is a finite probability that the block spin does not have the same sign as the majority of spins in the block:

| $\sum_{i\in I} \sigma_I$ | $\text{prob}(\mu = 1)_{\omega=1}$ | $\text{prob}(\mu = 1)_{\omega=2}$ |
|---|---|---|
| 4 | 0.999 664 65 | 0.999 999 89 |
| 2 | 0.982 013 79 | 0.999 664 65 |

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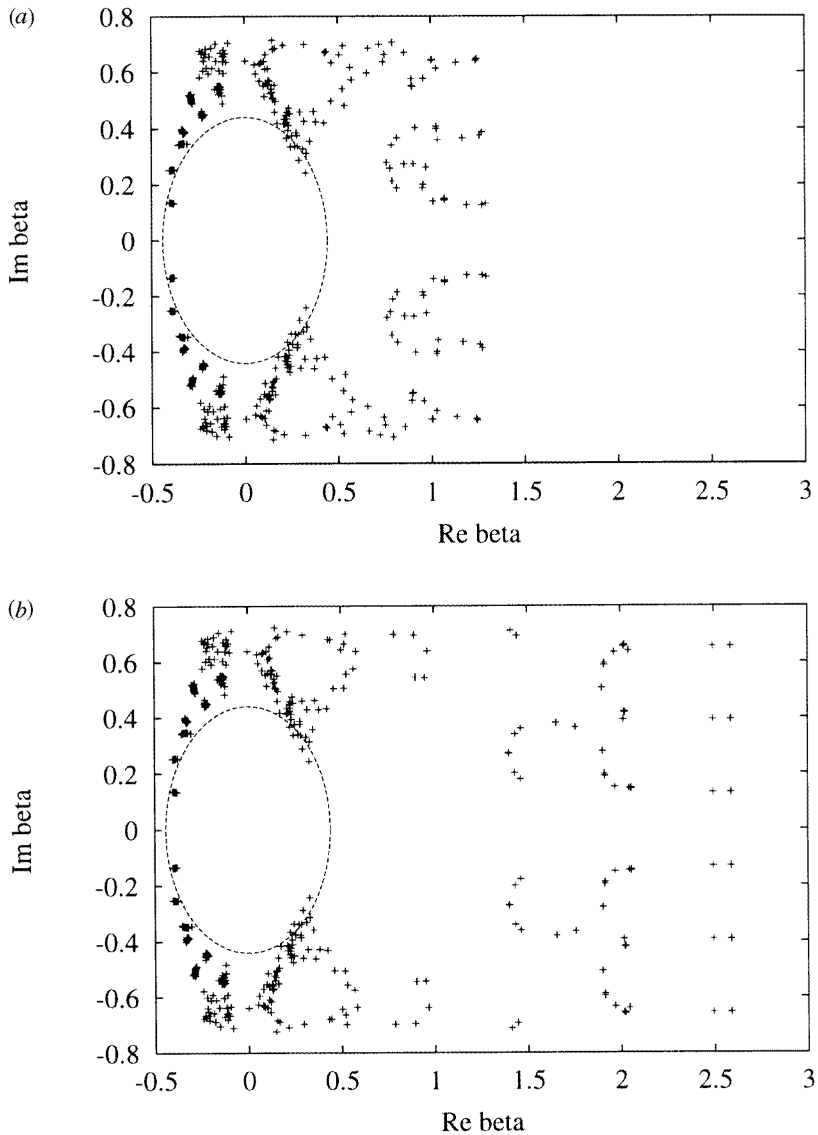


**Figure 6.** Zeros of $B(\mu)$ for $L = 6$, with Kadanoff rule. ($a$) $\omega = 1$, ($b$) $\omega = 2$.

In figures 5 and 6 we show the zeros for $L = 4$ and $L = 6$, both for $\omega = 1$ and $\omega = 2$. Obviously, most of the zeros in the neighbourhood of the critical circle do not move very much. In fact they are already very close to their majority rule values. However, compared with the $\omega = \infty$ case, extra zeros appear that populate the region of larger real part of $\beta$. It might be interesting to note that the zeros with the largest real part of $\beta$ belong to block spin configuration $c$#2, followed by those of $c$#1. Another observation, also clearly seen in figure 6, is that the spurious zeros move to the right when $\omega$ is increased. Most likely they are shifted to infinity when passing to the majority rule.

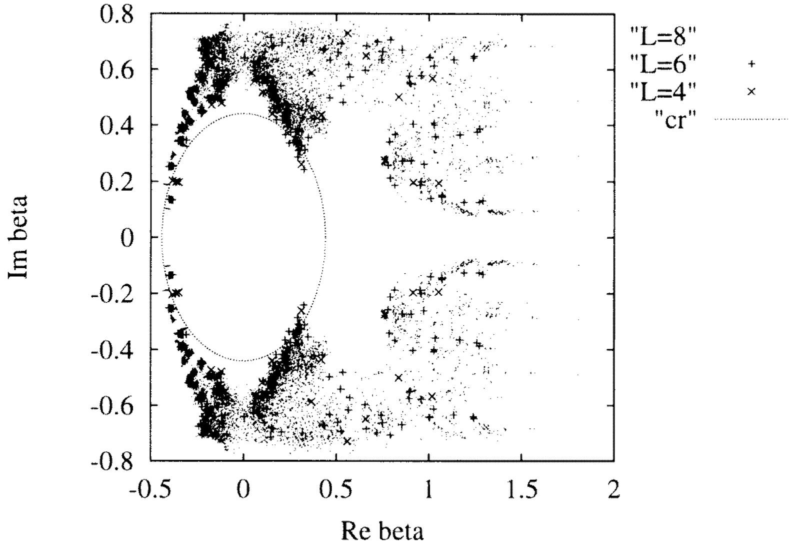


**Figure 7.** Zeros of $B(\mu)$ for $L = 4$, 6, and 8, with Kadanoff rule, $\omega = 1$.

In figure 7 the zeros of all three lattice sizes, $L = 4$, 6, and 8 are plotted for the case $\omega = 1$†. A careful inspection reveals that the zeros in the right half-plane *do* move towards the real axis when the lattice size is increased. They may well be reflecting the large $\beta$-pathologies.

## 6. Conclusions

The distribution of zeros of effective Boltzmannians was studied for 2D Ising systems. Both in the case of the majority and the Kadanoff rule a finite region around the critical point remains free of zeros, and when the volume is increased. In the case of the Kadanoff rule, however, zeros populate the right half-plane and approach the real axis at large $\beta$. They might be related to the pathologies of a number of Ising renormalization groups discussed in the literature. It would be very interesting to understand this relation (if it exists). Furthermore, one should try to understand the origin of the extra zeros that appear for finite $\omega$. The present study shows that the distribution of large $\beta$-zeros changes significantly when $\omega$ becomes finite. These findings do not, however, exclude the possibility of large $\beta$-pathologies of the 2D Ising majority rule renormalization group.

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† For $L = 8$, some 16 of the 479 block spin configurations were not taken into account because the search for the zeros of the corresponding polynomials suffered from numerical instabilities.

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